



# On the generating fields of Kloosterman sums

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Let  $f(x) \in \mathbb{F}_q[x]$  be a polynomial over a finite field with  $q = p^d$  elements, where p is a rational prime. Define the exponential sum

$$S_1(f) := \sum_{x \in \mathbb{F}_q} \zeta_p^{\operatorname{Tr}(f(x))} \in \mathbb{Z}[\zeta_p].$$

A basic problem is

- (1) as a complex number,  $|S_1(f)| = ?$
- (2) as a *p*-adic number,  $|S_1(f)|_p = ?$
- (3) as an algebraic number,  $\deg S_1(f) = ?$

# *L*-function

The first two questions have been studied extensively in the literature. Define

$$L(t,f) := \prod_{x \in \overline{\mathbb{F}}_p} \left( 1 - \operatorname{Tr}_{\mathbb{F}_q(x)/\mathbb{F}_p}(f(x)) t^{\deg x} \right)^{-1} = \exp\left(\sum_k S_k(f) \frac{t^k}{k}\right)$$

where 
$$S_k(f) := \sum_{x \in \mathbb{F}_{q^k}} \zeta_p^{\operatorname{Tr}(f(x))} \in \mathbb{Z}[\zeta_p].$$

Theorem (Dwork-Bombieri-Grothendick)

L(t, f) is a rational function.

Write

$$L(t, f) = \frac{\prod_{j} (1 - \beta_j t)}{\prod_{i} (1 - \alpha_i t)}.$$

Then

$$S_k(f) = \sum_i \alpha_i^k - \sum_j \beta_j^k.$$

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How to estimate the characteristic roots  $\alpha_i$  and  $\beta_j$ ? We need  $\ell$ -adic method. To describe it, let's recall the definition of sheaves.

Given a topological space X, there is a site Top(X) with

- (1) objects: the open subsets of X;
- (2) morphisms: the injection of open sets;
- (3) coverings: normal open coverings.

A sheaf  $\mathcal{F}$  on a topological space X over a field E is a contravariant functor  $\operatorname{Top}(X)^{\operatorname{op}} \to \operatorname{Vect}/E$ , which can be uniquely glued locally. That's to say, for any open covering  $U = \bigcup_i U_i$ ,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact.

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Let X be a scheme. Denote by  $X_{\mathrm{\acute{e}t}}$  the site with

- (1) objects: étale scheme  $X' \to X$ ;
- (2) morphisms: étale morphisms;
- (3) coverings:  $\{\varphi_i : X'_i \to X'\}$  with  $X' = \cup \varphi_i(X'_i)$ .

Fix a prime  $\ell \neq p$  and let E be a finite extension of  $\mathbb{Q}_{\ell}$ . An  $\ell$ -adic sheaf is a sheaf on  $X_{\text{\acute{e}t}}$  over E (which is constructible at every finite level).

Let K be c.d.v.f, with higher ramification groups  $I^{(r)}, r \ge 0$ . For any E-representation M of P, we have a decomposition  $M = \oplus M(x)$ , such that

$$M(0) = M^P, \quad M(x)^{I(x)} = 0, \quad M(x)^{I(y)} = M(x), \ y > x > 0.$$

We call x a break if  $M(x) \neq 0$ . Define

 $Sw(M) = \sum x \dim M(x).$ 

Let C be a proper smooth geometrically connected curve over a perfect field  $\mathbb{F}$ , with function field  $K = \mathbb{F}(C)$ . For any closed point  $x \in C(\mathbb{F})$ , we have the completion  $K_x$ .

For any non-empty open  $U \subset C$ , we have an equivalence of abelian categories

$$\{ \text{lisse } E \text{-sheaves on } U \} \longrightarrow \mathsf{Rep}^c_E \pi_1(U, \overline{\eta})$$

$$\mathcal{F} \longmapsto \mathcal{F}_{\overline{\eta}}.$$

Since  $\pi_1(U,\overline{\eta})$  is a quotient of  $\operatorname{Gal}(\overline{K}/K)$ , the decomposition group  $D_x \subset \operatorname{Gal}(\overline{K}/K)$ acts on  $\mathcal{F}_{\overline{\eta}}$ . We can define Swan conductor of  $\mathcal{F}$  at x. If  $x \in U$ , the action of  $I_x$  is trivial.

We will take  $\mathbb{F} = \mathbb{F}_p, C = \mathbb{P}^1$  and  $U = \mathbb{G}_m$ .

Assume that  $\mu_p \subseteq E$ . Deligne constructed a certain locally free of rank one  $\ell$ -adic sheaf  $\mathcal{F}_{\ell}(f)$  over E on  $\mathbb{G}_{a,\overline{\mathbb{F}}_p} = \operatorname{Spec} \overline{\mathbb{F}}_p[X]$ , such that

$$L(t, f) = \prod_{i} \det(1 - t \operatorname{Frob}, \operatorname{H}_{c}^{i})^{(-1)^{i+1}}$$

and

$$S_k(f) = \sum_i (-1)^i \operatorname{Tr}(\operatorname{Frob}^k, \operatorname{H}^i_c).$$

Here, Frob is the geometric Frobenius (inverse of  $\alpha \mapsto \alpha^p$ ),  $\mathrm{H}^i_c = \mathrm{H}^i_c(\mathbb{G}_{a,\overline{\mathbb{F}}_p}, \mathcal{F}_\ell(f))$  is the compact cohomology.

# $\ell$ -adic method, continue

Denote by  $\omega_{ij}$  the eigenvalues of Frob on  $\mathrm{H}_c^i$ , then

$$S_k(f) = \sum_{ij} (-1)^i \omega_{ij}^k.$$

Denote by  $B_i = \dim_E \mathrm{H}^i_c$  the Betti number.

### Theorem (Deligne)

 $\omega_{ij}$  is an algebraic integer and all its conjugates over  $\mathbb{Q}$  has same absolute value  $q^{r_{ij}/2}$ , where the weight  $0 \leq r_{ij} \leq i$  are integers.

Thus

$$|S_k| \leqslant \sum_i B_i q^{ki/2}.$$

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# **General case**

In general,

- (1) V a closed variety over  $\mathbb{F}_q$  of  $\mathbb{A}^N$ ,
- (2)  $\psi$  a non-trivial additive character on  $\mathbb{F}_q$ ,  $\psi_k = \psi \circ \operatorname{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}$ ,
- (3) f a regular function on V defined over  $\mathbb{F}_q$ ,
- (4)  $\chi$  a multiplicative character on  $\mathbb{F}_q^{ imes}$ ,  $\chi_k = \chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ ,
- (5) g an invertible regular function on V.

Define

$$S_k = \sum_{x \in V(\mathbb{F}_{q^k})} \psi_k(f(x)) \chi_k(g(x)).$$

Then Deligne's results still hold in this case. Moreover, Bombieri proved that the number of characteristic roots is at most

$$(4\max\{\deg V+1, \deg f\}+5)^{2N+1}.$$

Now we will consider

$$V = V(X_1 \cdots X_n - a), \quad f = X_1 + \cdots + X_n.$$

Let  $\chi = \{\chi_1, \ldots, \chi_n\}$  be an unordered *n*-tuple of multiplicative characters  $\chi_i : \mathbb{F}_q^{\times} \to \mu_{q-1}$ . Define the Kloosterman sum as

$$\mathrm{Kl}_{n}(\psi, \boldsymbol{\chi}, q, a) = \sum_{\substack{x_{1}\cdots x_{n}=a\\x_{i}\in\mathbb{F}_{q}}} \chi_{1}(x_{1})\cdots\chi_{n}(x_{n})\psi(\mathrm{Tr}_{\mathbb{F}_{q}/\mathbb{F}_{p}}(x_{1}+\cdots+x_{n})).$$

In this case, there are n characteristic roots with same weight n-1. Hence  $|\mathrm{Kl}_n| \leq nq^{(n-1)/2}$ .

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# **Galois** action

Clearly,  $\mathrm{Kl}_n \in \mathbb{Z}[\mu_{pc}]$ , where

$$c = \operatorname{lcm}_i \{ \operatorname{ord}(\chi_i) \}$$

divides q-1. Write

$$\operatorname{Gal}(\mathbb{Q}(\mu_{pc})/\mathbb{Q}) = \{\sigma_t \tau_w \mid t \in (\mathbb{Z}/p\mathbb{Z})^{\times}, w \in (\mathbb{Z}/c\mathbb{Z})^{\times}\},\$$

where

$$\sigma_t(\zeta_p) = \zeta_p^t, \quad \sigma_t(\zeta_c) = \zeta_c,$$
  
$$\tau_w(\zeta_p) = \zeta_p, \quad \tau_w(\zeta_c) = \zeta_c^w.$$

A basic observation tells

$$\sigma_t \tau_w \mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \prod \boldsymbol{\chi}(t)^{-w} \mathrm{Kl}_n(\psi, \boldsymbol{\chi}^w, q, at^n).$$

To study the generating fields of  $Kl_n$ , we need to consider the distinctness of different Kloosterman sums.

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When  $\chi = \mathbf{1} = \{1, \dots, 1\}$  is trivial, it's easy to see that

a, b conjugate  $\implies \operatorname{Kl}_n(\psi, \mathbf{1}, q, a) = \operatorname{Kl}_n(\psi, \mathbf{1}, q, b).$ 

When  $p > (2n^{2d} + 1)^2$  (Fisher), or  $p \ge (d-1)n + 2$  and p does not divide a certain integer (Wan), this is necessary. In general, it's conjectured that it's true when  $p \ge nd$ . Thus

$$\deg \operatorname{Kl}_n(\psi, \mathbf{1}, q, a) = \frac{p-1}{(p-1, n)}$$

under these conditions.

For our purpose, we need a different sheaf. Deligne and Katz defined the Kloosterman sheaf

$$\mathcal{K}l = \mathcal{K}l_{n,q}(\psi, \boldsymbol{\chi})$$

on  $\mathbb{G}_m \otimes \mathbb{F}_q = \operatorname{Spec} \mathbb{F}_q[X, X^{-1}]$ , with the following properties:

(1)  $\mathcal{K}$ l is lisse (locally constant at every finite level) of rank n and pure of weight n-1.

(2) For any 
$$a \in \mathbb{F}_q^{\times}$$
,  $\operatorname{Tr}(\operatorname{Frob}_a, \mathcal{K}l_{\overline{a}}) = (-1)^{n-1} \operatorname{Kl}_n(\psi, \chi, q, a)$ .

- (3)  $\mathcal{K}l$  is tame at 0 (Swan= 0).
- (4)  $\mathcal{K}l$  is totally wild with Swan conductor 1 at  $\infty$ . So all  $\infty$ -breaks are 1/n.

Fisher gave a descent of Kloosterman sheaves along an extension of finite fields. For any  $a \in \mathbb{F}_q^{\times}$ , he defined a lisse sheaf  $\mathcal{F}_a(\boldsymbol{\chi})$  on  $\mathbb{G}_m \otimes \mathbb{F}_p$ , such that  $\mathcal{F}_a(\boldsymbol{\chi}) | \mathbb{G}_m \otimes \mathbb{F}_q = \bigotimes_{\sigma \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)} (t \mapsto \sigma(a)t^n)^* \mathcal{K}l_n(\psi \circ \sigma^{-1}, \boldsymbol{\chi} \circ \sigma^{-1}).$ 

(1) *F<sub>a</sub>(χ)* is lisse of rank n<sup>d</sup> and pure of weight d(n − 1).
 (2) For any t ∈ 𝔽<sup>×</sup><sub>p</sub>, Tr(Frob<sub>t</sub>, *F<sub>a</sub>(χ)<sub>t</sub>)* = (−1)<sup>(n−1)d</sup>Kl<sub>n</sub>(ψ, χ, q, at<sup>n</sup>).
 (3) *F<sub>a</sub>(χ)* is tame at 0 and its ∞-breaks are at most 1.

#### Lemma

Let  $\mathcal{F}, \mathcal{F}'$  be lisse sheaves on  $\mathbb{G}_m \otimes \mathbb{F}_p$  of same rank r and pure of the same weight w. Assume that there is a root of unity  $\lambda$  such that for any  $t \in \mathbb{F}_p^{\times}$ , we have

 $\operatorname{Tr}(\operatorname{Frob}_t, \mathcal{F}_{\overline{t}}) = \lambda \operatorname{Tr}(\operatorname{Frob}_t, \mathcal{F}'_{\overline{t}}).$ 

Let  $\mathcal{G}$  be a geometrically irreducible sheaf of rank s on  $\mathbb{G}_m \otimes \mathbb{F}_p$ , pure of weight w, such that  $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$  occurs exactly once in  $\mathcal{F} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ . Then  $\mathcal{G} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$  occurs at least once in  $\mathcal{F}' \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ , provided that  $p > [2rs(M_0 + M_\infty) + 1]^2$ , where  $M_\eta$  is the largest  $\eta$ -break of  $\mathcal{F} \oplus \mathcal{F}'$ .

Assume not. Applying the Lefschetz Trace Formula to  $\mathcal{G}^\vee\otimes\mathcal{F}$  and  $\mathcal{G}^\vee\otimes\mathcal{F}',$  we have

$$\sum_{i=0}^{2} (-1)^{i} \operatorname{Tr}(\operatorname{Frob}, \operatorname{H}_{c}^{i}(\mathcal{G}^{\vee} \otimes \mathcal{F})) = \lambda \sum_{i=0}^{2} (-1)^{i} \operatorname{Tr}(\operatorname{Frob}, \operatorname{H}_{c}^{i}(\mathcal{G}^{\vee} \otimes \mathcal{F}')).$$

Apply Euler-Poincaré formula

$$h_c^0(\mathcal{F}) - h_c^1(\mathcal{F}) + h_c^2(\mathcal{F})$$
  
= rank  $\mathcal{F} \cdot \chi_c(\mathbb{G}_m \otimes \mathbb{F}_p) - \operatorname{Sw}_0(\mathcal{F}) - \operatorname{Sw}_\infty(\mathcal{F})$ 

to estimate  $Tr(Frob, H_c^1)$  (weight  $\leq 1$  by Weil II).

The *n*-tuple  $\chi$  is called *Kummer-induced* if there exsists a non-trivial character  $\Lambda$  such that  $\chi = \chi \Lambda := \{\chi_1 \Lambda, \dots, \chi_n \Lambda\}$  as unordered *n*-tuples. In this case,  $\prod \chi = \prod(\chi \Lambda) = \Lambda^n \prod \chi$  and thus  $\Lambda^n = 1$ .

Assume that p > 2n + 1 and  $\chi$  is not Kummer-induced. Then  $\mathcal{F}_a(\chi)$  has a highest weight with multiplicity one. Thus it has a subsheaf  $\mathcal{G}_a(\chi)$  such that, as representations of the Lie algebra  $\mathfrak{g}(\mathcal{F}_a(\chi))$ ,  $\mathcal{G}_a(\chi)$  is the irreducible sub-representation with highest weight. Moreover, it is geometrically irreducible and occurs exactly once in  $\mathcal{F}_a(\chi)$  over  $\mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ .

### Corollary

Let  $a, b \in \mathbb{F}_q^{\times}$  and let  $\chi$  and  $\rho$  be *n*-tuples of multiplicative characters  $\chi_i, \rho_j : \mathbb{F}_q^{\times} \to \overline{\mathbb{Q}}_\ell^{\times}$ . Assume that  $p > (2n^{2d} + 1)^2$ ,  $\chi$  is not Kummer-induced and

$$\operatorname{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \operatorname{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$$

for a fixed root of unity  $\lambda \in \mu_{q-1}$ . Then  $\mathcal{G}_a(\chi) \otimes \mathcal{L}_{\prod \overline{\chi}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$  occurs at least once in  $\mathcal{F}_b(\rho) \otimes \mathcal{L}_{\prod \overline{\rho}} \mid \mathbb{G}_m \otimes \overline{\mathbb{F}}_p$ .

Here  $\mathcal{L}_{\chi}$  is a rank one lisse sheaf on  $\mathbb{G}_m \otimes \mathbb{F}_p$  such that for  $t \in \mathbb{F}_p^{\times}$ ,

 $\operatorname{Tr}(\operatorname{Frob}_t, (\mathcal{L}_{\chi})_{\overline{t}}) = \chi(t).$ 

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Denote by

$$\mathcal{F} = \mathcal{F}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}, \ \mathcal{F}' = \mathcal{F}_b(\boldsymbol{\rho}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\rho}}}, \ \mathcal{G} = \mathcal{G}_a(\boldsymbol{\chi}) \otimes \mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}.$$

For  $t \in \mathbb{F}_p^{\times}$ , we have  $\sigma_t \lambda = \lambda$  and thus

$$(-1)^{(n-1)d} \operatorname{Tr}(\operatorname{Frob}_t, \mathcal{F}_{\overline{t}}) = \prod \overline{\chi}(t) \cdot \operatorname{Kl}_n(\psi, \chi, q, at^n)$$
$$= \sigma_t (\operatorname{Kl}_n(\psi, \chi, q, a)) = \lambda \sigma_t (\operatorname{Kl}_n(\psi, \rho, q, b))$$
$$= \lambda \prod \overline{\rho}(t) \cdot \operatorname{Kl}_n(\psi, \rho, q, bt^n) = (-1)^{(n-1)d} \lambda \operatorname{Tr}(\operatorname{Frob}_t, \mathcal{F}'_{\overline{t}}).$$

Apply Lemma to  $r = s = n^d, M_0 = 0, M_\infty \leq 1$ .

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### Distinctness

Now

$$\mathcal{G}_a(\boldsymbol{\chi})\otimes\mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}\hookrightarrow\mathcal{F}_b(\boldsymbol{
ho})\otimes\mathcal{L}_{\prod \overline{\boldsymbol{
ho}}}, \quad \mathcal{G}_b(\boldsymbol{
ho})\otimes\mathcal{L}_{\prod \overline{\boldsymbol{
ho}}}\hookrightarrow\mathcal{F}_a(\boldsymbol{\chi})\otimes\mathcal{L}_{\prod \overline{\boldsymbol{\chi}}}.$$

Thus the highest weight  $\lambda_a(\boldsymbol{\chi}) = \lambda_b(\boldsymbol{\rho})$ . Derived from this, and combining Fisher's arguments, we have:

Theorem (Z.)

Let  $a, b \in \mathbb{F}_q^{\times}$ . Assume that  $\chi, \rho$  are not Kummer-induced and neither of them is of type  $(\xi_1, \xi_1^{-1}, 1, \Lambda_2)\xi_2$ . If  $p > (2n^{2d} + 1)^2$  and

 $\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \lambda \mathrm{Kl}_n(\psi, \boldsymbol{\rho}, q, b)$ 

for some  $\lambda \in \mu_{q-1}$ , then there exists  $\sigma \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)$  and a multiplicative character  $\eta$ , such that  $b = \sigma(a)$  and  $\rho = \eta \cdot (\chi \circ \sigma^{-1})$  as unordered tuples. Moreover, either both Kloosterman sums vanish or  $\eta(b) = \lambda^{-1}$ .

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The last step is to show the non-vanishingness.

**Theorem** If  $p > (3n-1)C_{\chi} - n$  and for any  $i, j, \chi_i = \chi_j$  if  $\chi_i^n = \chi_j^n$ , then  $\mathrm{Kl}_n(\psi, \chi, q, a)$  is nonzero. Here  $C_{\chi} = \max_{i,j} \mathrm{lcm}(\mathrm{ord}(\chi_i), \mathrm{ord}(\chi_j))$  (1) is the supremum of least common multipliers of the orders of any two characters in  $\chi$ . We can express  $Kl_n$  as Gauss sums

$$(q-1)\mathrm{Kl}_n(\psi, \boldsymbol{\chi}, q, a) = \sum_{m=0}^{q-2} \omega^m(a) \prod_{i=1}^n g(m+s_i)$$

by Fourier transform on  $\mathbb{F}_q^{\times}$ , where  $\chi_i = \omega^{s_i}$  for a Teichmüller character. What we need to do is to proof there is a unique m such that the valuation of  $\prod_{i=1}^n g(m+s_i)$  is minimal.

#### Theorem (Z.)

If  $p > \max\left\{(2n^{2d}+1)^2, (3n-1)C_{\chi}-n\right\}$  and for any  $i, j, \chi_i = \chi_j$  if  $\chi_i^n = \chi_j^n$ , then  $\mathrm{Kl}_n(\psi, \chi, q, a)$  generates  $\mathbb{Q}(\mu_{pc})^H$ , where H consists of those  $\sigma_t \tau_w$  such that there exists an integer  $\beta$  and a character  $\eta$  satisfying

$$t = \lambda a_1^{\beta}, \lambda^{n_1} = 1, \ \boldsymbol{\chi}^w = \eta \boldsymbol{\chi}^{q_1^{\beta}}, \ \eta(a) = \prod \boldsymbol{\chi}^w(t).$$

Here  $n_1 = (n, p - 1)$ ,  $q_1 = \# \mathbb{F}_p(a^{(p-1)/n_1})$  and  $a_1 \in \mathbb{F}_p^{\times}$  such that  $a_1^{n/n_1} = N_{\mathbb{F}_{q_1}/\mathbb{F}_p}(a^{(1-p)/n_1}) = a^{(1-q_1)/n_1}$ .

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### An example: n = 2 case

Let  $\chi = \{1, \chi\}$ , where  $\chi$  is a multiplicative character of order  $c \neq 2$ . If  $p > \max\left\{(2^{2d+1}+1)^2, 5c-2\right\}$ , then  $\mathrm{Kl}(\psi, \chi, p^d, a)$  generates  $\mathbb{Q}(\mu_{pc})^H$ , where

$$H = \begin{cases} \langle \tau_{q_1} \sigma_{a_1}, \sigma_{-1}, \tau_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a) = 1; \\ \langle \tau_{-q_1} \sigma_{a_1}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a) = \chi(a_1) = -1; \\ \langle \tau_{q_1}^{\alpha} \sigma_{a_1}^{\alpha}, \sigma_{-1} \rangle, & \text{if } \chi(-1) = 1, \chi(a)^{\alpha} \neq 1; \\ \langle \tau_{q_1} \sigma_{-a_1}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = \chi(a_1) = -1; \\ \langle \tau_{q_1} \sigma_{a_1}, \tau_{-1} \rangle & \text{if } \chi(-1) = -1, \chi(a) = 1; \\ \langle \tau_{q_1} \sigma_{a_1}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = -1, \chi(a_1) = 1; \\ \langle \tau_{q_1} \sigma_{a_1}, \tau_{-1} \sigma_{-1} \rangle, & \text{if } \chi(-1) = -1, \chi(a) = -1, \chi(a_1) = 1; \\ \langle \tau_{q_1}^{\alpha} \sigma_{a_1}^{\alpha} \rangle, & \text{if } \chi(-1) = -1, 2 \mid \alpha, \chi(a) \neq \pm 1. \end{cases}$$

 $q_1 = \#\mathbb{F}_p(a^{(1-p)/2}), a_1 = a^{(1-q_1)/2}$  and  $\alpha$  is the order of  $\chi(a_1) \in \mu_{p-1}$ .

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### Remark

Consider the Kloosterman sums

$$S_k = \operatorname{Kl}(\psi, \boldsymbol{\chi} \circ \boldsymbol{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}, q^k, a).$$

If  $p > \max\left\{(2n^{2dk} + 1)^2, (3n - 1)C_{\chi} - n\right\}$ , then  $\mathbb{Q}(S_k) = \mathbb{Q}(\mu_{pc})^H$ , where H consists of those  $\sigma_t \tau_w$  such that there exists an integer  $\beta$  and a character  $\eta$  on  $\mathbb{F}_q^{\times}$  satisfying

$$t = \lambda a_1^{\beta}, \lambda^{n_1} = 1, \quad \boldsymbol{\chi}^w = \eta \boldsymbol{\chi}^{q_1^{\beta}}, \quad \eta(a) = \gamma \cdot \prod \boldsymbol{\chi}^w(t), \gamma^k = 1$$

Thus  $\mathbb{Q}(S_k) = \mathbb{Q}(S_{k-c})$  since  $\gamma^c = 1$ .

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The L-function

$$L(T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} S_k\right)$$

is a rational function. Thus the sequence  $\{S_k\}_k$  is linear recurrence sequence. The sequence  $\{\mathbb{Q}(S_k)\}_{k \ge N}$  is periodic of period r for some N (Wan, Yin). Thus if  $p > \max\left\{\left(2n^{2d(N+r)}+1\right)^2, (3n-1)C_{\chi}-n\right\}$ , the generating field of  $S_k$  is determined by the previous equations for any k. For this purpose, we need to decrease the bound  $(2n^{2d}+1)^2$  and estimate the period r and N. We conjecture that  $S_k$  has the predicted generating field if p > 3ndc.

